On Approximations by *- Ideals in a Ring with Involution

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Abstract:

The aim of this paper is to present the concepts of * - ring approximation spaces, the congruence relation, * - ideal in a ring, and the lower and upper approximations of any subset of a ring with involution respect to * -ideals. Some properties of approximation operators are discussed. We introduce the rough * - ideals in a ring with involution supporting it with some theorems and illustrative examples.

***Keywords*:** lower approximation, upper approximation, ideals, involution ring.

الملخص:

الهدف من هذه الورقة هو تقديم المفاهيم الأتية: الفضاء التقريبي الحلقي الالتفافي، التطابق الالتفافي والمثالية الالتفافية في الحلقات. كذلك دارسة التقريبان السفلي والعلوي لأي مجموعة جزئية من الحلقة الالتفافية بالنسبة للمثالية الالتفافية. تمت مناقشة بعض خصائص عوامل التقريب. أيضا نقدم مثاليات تقريبية التفافية مدعمين كل تلك المفاهيم ببعض النظريات والامثلة التوضيحية.

1. Introduction:

Pawlak introduced the theory of rough sets in 1982 [1]. It is an independent method to deal the vagueness and uncertainty. It is an extension of the set theory, in which a pair of ordinary sets called the lower and upper approximations, describes a subset of a universe. Pawlak used an equivalence classes for the construction of lower and upper approximations of a set. It soon invoked a natural question concerning possible connection between the rough sets and an algebraic system. Many researchers studied the algebraic approach of rough sets in different algebraic structures such as [2,3,4,5]. Biswas and Nanda [6] introduced the notion of rough subgroups. Mordeson [7] applied a rough set theory to the fuzzy ideal theory. Some concepts of lattice in a rough set theory has studied by Yao [8]. Chinram [9] studied the rough prime ideas and the rough fuzzy prime ideals in Gamma-semigroups. Kuroki in [10,11] introduced the notion of rough ideals of a semigroup. Davvaz in [12] introduced the notion of rough subring with respect to ideals of a ring. Abdunabi in [13] introduced the connection between a rough set theory and a ring theory.

Rings with involution have been studied in [14,15], where if $\mathfrak{R}$ be a ring then an additive map $x \longmapsto x^*$ of $\mathfrak{R}$ into itself is called an involution if:

(i) $(x+y)^* = x^* + y^*$ (ii) $(xy)^* = y^* x^*$ (iii) $(x^*)^* = x$
hold for all $x, y \in \mathfrak{R}$.

In this paper, we present a concept of * - ring approximation spaces by using the involution ring. We introduce the lower and upper approximations of any subset of these spaces with respect to * -ideals and discuss some properties of the approximation operators. In addition, the rough * - ideals in * - ring approximation spaces are studied. These newly introduced concepts have supported by some examples and theorems that highlights its utility and future applicability.

2. Pawlak approximation space:

In this section, some well-known basic identities are given; which will used extensively in the forthcoming sections. Suppose U is a non–empty set. A partition to classification of U is a family of the non-empty subsets of U such that each element of U is contained in exactly one element of R. Recall that an equivalence relation R on a set U is a reflexive, symmetric, and transitive binary relation on U. Each partition R induces an equivalence relation R on U by setting:

$x\ R\ y \Leftrightarrow x$ and y are in the same class R.

Conversely, each equivalence relation R on U induces a partition R of U whose classes have the form $[x]_R = \{y \in U: x\ R\ y\}$.

Definition 2.1[1]: A pair (U, R) where $U \neq \emptyset$ and R is an equivalence on U is called the Pawlak approximation space.

Definition 2.2 [1]: For an approximation space (U, R) and $R: P(U) \to P(U) \times P(U)$: For every $X \in P(U)$; $X \subseteq U$. We can approximate X as: $R\ (X) = (\underline{R}\ (X), \overline{R}\ (X))$, where $\underline{R}\ (X) = \{x \in U: [x]_R \subseteq X\}$, $\overline{R}\ (X) = \{x \in U: [x]_R \cap X \neq \emptyset\}$. $\underline{R}\ (X)$ is called the lower approximation of X and $\overline{R}\ (X)$ is called the upper approximation of X in (U, R) respectively.

Clearly; $\underline{R}\ (X)$is the set of all objects which can be with certainty classified as members of X with respect to R and $\overline{R}\ (X)$ is the set of all objects which can be only classified as possible members of X with respect to R.

Definition 2.3 [1]: Let (U, R) be an approximation space, $X \subseteq U$. We say:

(i) X is a rough (undefinable) set if $\underline{R}\ (X) \neq \overline{R}\ (X) \neq X$.

(ii) X is an exact (definable) set if $\underline{R}\ (X) = \overline{R}\ (X) = X$.

Definition 2.4 [1]: Let (U, R) be an approximation space, the following areas can be defined:

(i) The boundary region of X is define by $BX_R = \overline{R}\ (X) - \underline{R}\ (X)$

(ii) The internal edge of X is given by: : $\underline{ED}\ (X) = X - \underline{R}\ (X)$.

(iii) The external edge of X is given by: $\overline{ED}\ (X) = \overline{R}\ (X) - X$.

The boundary region of X is the set of all objects which can be decisively classified neither as members of X nor as members of X^c with respect to R.

Corollary: 2.1 Let $A = (U, R)$ be an approximation space and $X \subseteq U$. Then:

(i) X is definable set if and only if $BX_R = \emptyset$.

(ii) X is rough set if and only if $BX_R \neq \emptyset$.

Proposition 2.3 [1]**:** Let $X, Y \subseteq U$, where U is a universe and X^c denoted the complementation of X in U, then the a approximations have the following properties:

(i) $\underline{R\,X} \subseteq X \subseteq \overline{RX}$.

(ii) $\underline{R\,\emptyset} = \overline{R\emptyset}\ ,\ \underline{R\,U} = \overline{RU}$.

(iii) $\underline{R(X \cup Y)} \supseteq \underline{R\,X} \cup \underline{R\,Y}$.

(iv) $\underline{R\,(X \cap Y)} = \underline{R\,X} \cap \underline{R\,Y}$.

(v) $\overline{R(X \cup Y)} = \overline{RX} \cup \overline{RY}$.

(vi) $\overline{R(X \cap Y)} \subseteq \overline{RX} \cap \overline{RY}$.

(vii) $\overline{RX^c} = \underline{(R\,X)^c}$.

(viii) $\underline{R\,X^c} = \overline{(R\,X)^c}$.

(ix) $\underline{R(\underline{R\,X})} = \overline{R\,(\underline{R\,X})} = \underline{R\,X}$.

(x) $\overline{R(\overline{RX})} = \underline{R(\overline{RX})} = \overline{RX}$.

Example 2.1: Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ the equivalence relation R is defined as: $R = \{\{x_1\}, \{x_5\}, \{x_2, x_4\}, \{x_3, x_6\}\}$. If $X = \{x_1, x_5, x_6\}$, $Y = \{x_2, x_4\}$. Then $\underline{R\,(X)} = \{x_1, x_5\}$, $\overline{R(X)} = \{x_1, x_3, x_5, x_6\}$ and $B(X_R) \neq$

$\emptyset$. So X is rough set. $\underline{R(Y)} = \overline{R(Y)} = \{x_2, x_4\} = Y$ and $B(Y_R) = \emptyset$. So Y is definable set.

3. Ring approximation spaces:

Here, we introduce some basic concepts for the sake of completeness. Recall from [12].

Definition 3.1: A non-empty set $\mathfrak{R}$ with two binary operations + (addition) and • (multiplication) called a ring if it satisfies the following axioms:

(i) $(\mathfrak{R}, +)$ is an additive group.
(ii) $(\mathfrak{R}, \bullet)$ is a semigroup;
(iii) $(a_1 + a_2) \bullet a_3 = a_1 \bullet a_3 + a_2 \bullet a_3$, and $a_1 \bullet (a_2+a_3) = a_1 \bullet a_2 + a_1 \bullet a_3$ for all $a_1, a_2, a_3 \in \mathfrak{R}$.

Definition 3.2: A subset I of a ring $\mathfrak{R}$ is called a left (resp. right) ideal of $\mathfrak{R}$ if it satisfies the condition ($aI \subseteq I$ ($Ia \subseteq I$) for $a \in \mathfrak{R}$.

Clearly a left (resp. right) ideal of $\mathfrak{R}$ is a subring of $\mathfrak{R}$. A two sides ideals of a ring $\mathfrak{R}$ (briefly called an ideal of $\mathfrak{R}$) is both a left and a right ideal of $\mathfrak{R}$.

Definition3.3: Let I be an ideal of a ring $\mathfrak{R}$. For $a, b \in \mathfrak{R}$ then

$a \equiv b\ (mod\ A)\ if\ a - b \in I$ … ….. (1)

We say a is congruent to b mod A. It easy to see the relation (1) is an equivalence relation. So the pair ($\mathfrak{R}$, mod A) is an approximation space. We shall called the pair ($\mathfrak{R}$, mod A) is a ring approximation space.

Definition3.4: Let I be an ideal of $\mathfrak{R}$ and X be a non-empty subset of a ring approximation space ($\mathfrak{R}$, mod A). The lower and the upper approximations of X are defined respectively with respect to the ideal I as follows:

$\underline{I(X)} = \cup \{x \in \Re : (x + I) \subseteq X\}$, $\overline{I(X)} = \cup \{x \in \Re : (x + I) \cap X \neq \emptyset\}$.

Definition 3.5: let $(\Re, \bmod A)$ be a ring approximation space, the following areas can be defined:

(i) The boundary region of X with respect of I is define by $B_{\mathrm{I}}(X) = \overline{I(X)} - \underline{I(X)}$.

(ii) The internal edge of X with respect of I is given by: $\underline{ED_I}(X) = X - \underline{I(X)}$.

(iii) The external edge of X with respect of I is given by: $\overline{ED_I}(X) = \overline{I(X)} - X$.

If $B_I(X) \neq \emptyset$, we say X is rough set with respect of I. However, if $B_I(X) = \emptyset$, we say X is definable set with respect of I.

Proposition 3.1: Let I be an ideal of a ring $\Re$, and X is a rough set with respect to I, we have:

(i) If **)** $\underline{I(X)}$ and $\overline{I(X)}$ are ideals of $\Re$, then X is a rough ideal.

(ii) If $\underline{I(X)}$ and $\overline{I(X)}$ are subring of $\Re$, then X is a rough ring.

Remark 3.1: Let I be an ideal of a ring $\Re$, and X is rough set with respect to I,

(i) If **)** $\underline{I(X)}$ and $\overline{I(X)}$ are not ideals of R, then X is not a rough ideal.

(ii) (ii) If **)** $\underline{I(X)}$) and $\overline{I(X)}$are not subring of $\Re$, then X is not a rough ring.

The following example shows remark 3.1.

Example 3.1: We consider the ring $\Re = Z_8$ and the ideal $I = \{0, 2, 4, 6\}$ is the only ideal in Z_8. Let $X = \{1, 2, 3, 4, 5, 6, 7\} \subseteq Z_8$. Then, for $x \in \Re$ we calculate $x + I = \{0, 2, 4, 6\}, \{1, 3, 5, 7\}$. Since $\underline{I}(X) = \cup \{x \in \Re: (x + I) \subseteq X\}$, Then $\underline{I}(X) = \{1, 3, 5, 7\}$ but not ideal because $\forall\, x \in Z_8 \wedge \forall\, r \in I(X)$ we find that $0 \in Z_8 \wedge 3 \in \overline{I(X)}$. So $0 .3 = 0 \notin \underline{I}(X)$. As well $\underline{I}(X)$is not subring because $3 - 1 = 2 \notin \underline{I}(X)$. Also $\overline{I(X)} = \cup \{x \in \Re: (x + I) \cap X \neq \emptyset\} = \{0, 2, 4, 6\} \cup \{1, 3, 5, 7\} = Z_8$ is an ideal and subring. So X is not neither a rough ideal nor a rough ring.

4. * - Ring Approximation Spaces:

In this section, we introduce the concepts of $*$ - ring approximation spaces, the lower and the upper approximations of a non – empty subset of the involution ring with respect of * - ideal. In addition, we study some properties of these approximations.

Definition 4.1[15]: A ring $\Re$ is said to be an involution ring (* - Ring) and denoted by $\Re^*$ if there is defined an involution * subject to the identities:

$(a + b)^* = a^* + b^*$, $(ab)^* = b^* a^*$, $a^{**} = a$ for all $a, b \in \Re^*$. If $\Re$ is commutative then the identical mapping of $\Re$ onto $\Re$ is an involution on $\Re$.

Definition 4.2[14]: An ideal I of an involution ring $(\Re^*)$ is called $*$ - ideal, and denoted by I^*, if it is closed under involution; that is: $I^* = \{a^* \in \Re^*; a \in I\} \subseteq I$.

In the theory of involution rings, $*$ - ideals has been used successfully (instead of one–sided which make no sense in describing their structure (see [14] and [15]).

Definition 4.3: Let $\Re^*$ is an involution ring. Then we call the pair $(\Re^*, \bmod A)$ is a $*$ - ring approximation space.

Definition 4.4: let $(\Re^*, \text{mod } A)$ be a * - ring approximation space, I^* is an $*$ - ideal. We can redefine the lower and the upper approximations of X with respect of I^* as:

$\underline{I^*(X)} = \cup \{x^* \in \Re^* : (x^* + I^*) \subseteq X\}$, $\overline{I^*(X)} = \cup \{x^* \in \Re^* : (x^* + I^*) \cap X \neq \emptyset\}$, . Where $X \subseteq \Re^*$.

Definition 4.5: let $(\Re^*, \text{mod } A)$ be a * - ring approximation space, the following regions can be defined:

(i) The boundary region of X with respect of I^* is given by: $B_{I^*}(X) = \overline{I^*(X)} - \underline{I^*(X)}$.

(ii) The internal edge of X with respect of I^* is given by: $\underline{ED_{I^*}}(X) = X - \underline{I^*(X)}$.

(iii)The external edge of X with respect of I^* is given by: $\overline{ED_{I^*}}(X) = \overline{I^*(X)} - X$.

Proposition 4.1: If $B_{I^*}(X) \neq \emptyset$, then X is rough set with respect of I^*. Otherwise, X is definable.

For the * - ring approximation space $(\Re^*, mod\ A)$. The rough (undefinable) set can be expressed by its approximations with respect to I^* and written in the following form:

$Apr(X) = (\underline{I^*(X)}, \overline{I^*(X)})$; $X \subseteq \Re^*$.

The following example shows definition 4.5.

Example 4.1: Let a ring $\Re^* = Z_6$ and the involution is defined by $a^* = a \ \forall a \in Z_6$.

Let $I^* = \{0, 2, 4\}$ is a * - ideal, $X = \{1, 2, 3, 4, 5\}$ and $Y = \{1, 3, 5\}$.

It is clearly $I^* = \{0^*, 2^*, 4^*\} = \{0, 2, 4\} = I$.

For $x^* \in \Re^* : x^* + I^* = x + I$, We calculate $x^* + I^*$: $0^* + I^* = 2^* + I^* = 4^* + I^* = \{0^*, 2^*, 4^*\} = \{0, 2, 4\}$, $1^* + I^* = 3^* + I^* = 5^* + I^* = \{1^*, 3^*, 5^*\} = \{1, 3, 5\}$. So $\underline{I^*(X)} = \{x^* \in \Re^* : (x^* + I^*) \subseteq X\} = \{1,3,5\}$ and

$\overline{I^*(X)} = \cup\, \{x^* \in \Re^* : (x^* + I^*) \cap X \neq \emptyset\} = \{0,2,4\} \cup \{1,3,5\} = \{0, 1, 2, 3, 4, 5\}$.

$\underline{I^*(Y)} = \overline{I^*(X)} = \{1,3,5\}$. Therefore, Y is definable set with respect of I^* while X is rough. Also, $B_{I^*}(X) = \overline{I^*(X)} - \underline{I^*(X)} = \{0, 2, 4\}$. $\underline{ED_{I^*}}(X) = X - \underline{I^*(X)} = \{2, 4\}$.

$\overline{ED_{I^*}}(X) = \overline{I^*(X)} - X = \{0\}$.

In a similar way we can get $B_{I^*}(Y)$, $\underline{ED_{I^*}}(Y)$, $\overline{ED_{I^*}}(Y)$ for a set Y.

Corollary 4.1: For a * - ring approximation space ($\Re^*$, mod A) and I^* be $*$ - ideal. Then

(i) $\underline{I^*(X)}$, $\overline{I^*(X)}$ are definable sets for every $X \subseteq \Re*$.

(i) For every $x \in \Re^*$, $x + I^*$ is definable set.

Proof: It is directly.

We can get the properties of approximation operators for any subset of a * - ring approximation space ($\Re^*$, $modA$) in the following proposition.

Proposition 4.2: For a* - ring approximation space ($\Re^*$, $modA$), I^* be an $*$ - ideal. Let $A, B \subseteq \Re^*$ we have:

(i) $\underline{I^*(A)} \subseteq A \subseteq \overline{I^*(A)}$.

(ii) $\underline{I^*(\emptyset)} = \emptyset = \overline{I^*(\emptyset)}$.

(iii) $\underline{I^*(\Re^*)} = \Re^* = \overline{I^*(\Re^*)}$.

(iv) $\underline{I^*(A \cap B)} = \underline{I^*(A)} \cap \underline{I^*(B)}$.

$(v)\ \underline{I^*(A \cup B)} \supseteq \underline{I^*(A)} \cup \underline{I^*(B)}$.

$(vi)\ \overline{I^*(A \cup B)} = \overline{I^*(A)} \cup \overline{I^*(B)}$.

(vii) $\overline{I^*(A \cap B)} \subseteq \overline{I^*(A)} \cap \overline{I^*(B)}$.

(viii) If $A \subseteq B$, *then* $\underline{I^*(A)} \subseteq \underline{I^*(B)}$ *and* $\overline{I^*(A)} \subseteq \overline{I^*(B)}$.

(ix) $\underline{I^*(\underline{I^*(A)}} = \overline{I^*\ (\underline{I^*(A)}} = \underline{I^*(A)}$.

$(x)\ \overline{I^*(\overline{I^*(A)})} = \underline{I^*(\overline{I^*(A)})} = \overline{I^*(A)}$.

$(xi)\ \underline{I^*((A^c)} = \left(\overline{I^*(A}\right)^c$.

(xii) $\overline{I^*((A^c)} = (\underline{I^*(A)})^c$.

Proof:

(i) Let $x^* \in \underline{I^*(A)}$; $\underline{I^*(A)} = \{x^* \in \mathfrak{R}^* : (x^* + I^*) \subseteq A\}$ then $x^* \in x^* + I^* \subseteq A \implies \underline{I^*(A)} \subseteq A$. And so let $x^* \in A$ since $x^* \in x^* + I^*$ then $x^* \in (x^* + I^*) \cap \mathrm{A} \implies (x^* + I^*) \cap A \neq \emptyset$. So $x^* \in \overline{I^*(A)}$; $\overline{I^*(A)} = \{x^* \in \mathfrak{R}^* : (x^* + I^*) \cap A \neq \emptyset\}$
Subsequently; $\underline{I^*(A)} \subseteq A \subseteq \overline{I^*(A)}$.

(ii) $\underline{I^*(\emptyset)} = \{x^* \in \mathfrak{R}^* : (x^* + I^*) \subseteq \emptyset\} = \emptyset = \{x^* \in \mathfrak{R}^* : (x^* + I^*) \cap \emptyset = \emptyset\} = \overline{I^*(\emptyset)}$.

There for $\underline{I^*(\emptyset)} = \emptyset = \overline{I^*(\emptyset)}$.

(iii) It says the way in (i).
$\underline{I^*\ (\mathfrak{R}^*)} = \{x^* \in \mathfrak{R}^* : (x^* + I^*) \subseteq \mathfrak{R}^*\} = \mathfrak{R}^* = \{x^* \in \mathfrak{R}^* : (x^* + I^*) \cap \mathfrak{R}^* \neq \emptyset\} = \overline{I^*(\mathfrak{R}^*)}$.

(iv) Let $x^* \in \underline{I^*(A \cap B)} = \{x^* \in \Re^* : (x^* + I^*) \subseteq A \cap B\} \Leftrightarrow x^* + I^* \subseteq A \wedge x^* + I^* \subseteq B \Leftrightarrow x^* \in \underline{I^*(A)} \wedge x^* \in \underline{I^*(B)} \Leftrightarrow x^* \in \underline{I(A)} \cap \underline{I(B)}$.

(v) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$ then $\underline{I^*(A)} \subseteq \underline{I^*(A \cup B)} \vee \underline{I^*(B)} \subseteq \underline{I^*(A \cup B)}$. So $\underline{I^*(A)} \cup \underline{I^*(B)} \subseteq \underline{I^*(A \cup B)}$.

(vi) and (vii) It says the way in (iv), (v) respectively.

(viii) Let $x^* \in \overline{I^*(A)}$; $\overline{I^*(A)} = \{x^* \in \Re^* : (x^* + I^*) \cap A \neq \emptyset\}$ and Since $A \subseteq B$ then $(x^* + I^*) \cap B \neq \emptyset$. So $x^* \in \overline{I^*(B)}$; $\overline{I^*(\mathrm{B})} = \{x^* \in \Re^* : (x^* + I^*) \cap B \neq \emptyset\}$

Subsequently; $\overline{I^*(A)} \subseteq \overline{I^*(B)}$. In a similar way we can prove $\underline{I^*(A)} \subseteq \underline{I^*(B)}$.

(ix) $\underline{I(\underline{I(A)})} = \left\{x^* \in \Re^* : (x^* + I^*) \subseteq \underline{I^*(A)}\right\} = \{x^* \in \Re^* : (x^* + I^*) \subseteq \{x^* \in \Re^* : (x^* + I^*) \subseteq A\}\} = \{x^* \in \Re^* : (x^* + I^*) \subseteq A\} = \underline{I^*(A)}$. And so $\overline{I^*(\underline{I^*(A)})} = \left\{x^* \in \Re^* : (x^* + I^*) \cap \underline{I^*(A)} \neq \emptyset\right\} = \{x^* \in \Re^* : (x^* + I^*) \cap \{x^* \in \Re^* : (x^* + I^*) \subseteq A\} \neq \emptyset\} = \{x^* \in \Re^* : (x^* + I^*) \subseteq A\} = \underline{I^*(A)}$.

(x) It says the way in (ix).

(xi) $\underline{I^*(A)} = \{x^* \in \Re^* : (x^* + I^*) \subseteq A\} = \{x^* \in \Re^* : (x^* + I^*) \subseteq (A^c)^c\} = \underline{\mathrm{I}^*((A^c)^c)} = \{x^* \in \Re^* \cap (A^c)^c \neq \emptyset\} = (\overline{I^*(A^c})^c$.

(xii) It same the way in (xi) ■

The following example shows proposition 4.2.

Example 4.2: Let the ring $\Re = \mathrm{Z}_{12}$ and define the involutionon $\Re$ by a * = a ∀ a ∈ Z_{12}.

Then $\Re^* = Z_{12}$. Now Suppose a $*-$ ideal is $\{0, 6\}$ in $\Re^*$ and let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, $B = \{1, 2, 4, 6, 8, 10\}$, $C = \{5, 7, 9, 11\}$. Since $A \cap B = \{1, 2, 4, 6, 8, 10\}$, $A \cap C = C$, $B \cap C = \emptyset$ Then for $x^* \in Z_{12}$ we calculate $x^* + I^* = \{0, 6\}, \{1, 7\}, \{2, 8\}, \{3, 9\}, \{4, 10\}, \{5, 11\}$

Now, we can calculate the properties of approximation operators for some subsets.

$$\underline{I^*(\emptyset)} = \{x^* \in Z_{12}: (x^* + I^*) \subseteq \emptyset\} = \emptyset = \{x^* \in Z_{12}: (x^* + I^*) \cap \emptyset \neq \emptyset\} = \overline{I^*(\emptyset)}$$

$$\underline{I^*(Z_{12})} = \{x^* \in Z_{12}: (x^* + I^*) \subseteq Z_{12}\} = Z_{12}$$
$$= \{x^* \in Z_{12}: (x^* + I^*) \cap Z_{12} = \emptyset\} = \overline{I^*(Z_{12})}$$

$$\underline{I^*(A)} = \{x^* \in Z_{12}: (x^* + I^*) \subseteq A\} = \{1,7\} \cup \{2,8\} \cup \{3,9\} \cup \{4,10\} \cup \{5,11\} = \{1,2,3,4,5,6,7,8,9,10,11\},$$

$$\underline{I^*(B)} = \{x^* \in Z_{12}: (x^* + I^*) \subseteq B\} = \{2,8\} \cup \{4,10\} = \{2,4,8,10\}$$

$$\underline{I^*(C)} = \{x^* \in Z_{12}: (x^* + I^*) \subseteq C\} = \{5,11\},$$

$$\underline{I^*(A \cap B)} = \{x^* \in Z_{12}: (x^* + I^*) \subseteq (A \cap B)\} = \{2,8\} \cup \{4,10\}$$
$$= \{2,4,8,10\},$$

$$\underline{I^*(A)} \cap \underline{I^*(B)} = \{2,4,8,10\} = \underline{I^*(A \cap B)},$$

$$\overline{I^*(A)} = \{x^* \in Z_{12}: (x^* + I^*) \cap A \neq \emptyset\} = \{0,6\} \cup \{1,7\} \cup \{2,8\} \cup \{3,9\} \cup \{4,10\} \cup \{5,11\} = Z_{12},$$

$$\overline{I^*(B)} = \{x^* \in Z_{12}: (x^* + I^*) \cap B \neq \emptyset\} = \{0,6\} \cup \{1,7\} \cup \{2,8\} \cup \{4,10\} = \{0, 1, 2, 4, 6, 7, 8, 10\},$$

$$\overline{I^*(C)} = \{x^* \in Z_{12}: (x^* + I^*) \cap B \neq \emptyset\} = \{1,7\} \cup \{3,9\} \cup \{5,11\} = \{1,3,5,7,9,11\},$$

$\overline{I^*(A \cap B)} = \{x^* \in Z_{12}: (x^* + \mathrm{I}^*) \cap (\mathrm{A} \cap \mathrm{B}) \neq \emptyset\} = \{0, 1, 2, 4, 6, 7, 8, 10\}$,

$\overline{I^*(A)} \cap \overline{I^*(B)} = \{0, 1, 2, 4, 6, 7, 8, 10\} = \overline{I^*(A \cap B)}$,

$\underline{I^*(A \cup B)} = \{x^* \in Z_{12}: (x^* + I^*) \subseteq (A \cup B)\} = Z_{12}$,

$\underline{I^*(A)} \cup \underline{I^*(B)} = Z_{12} = \underline{I^*(A \cup B)}$,

$\overline{I^*(A \cup B)} = \{x^* \in Z_{12}: (x^* + I^*) \cap (A \cup B) \neq \emptyset\} = \{0, 6\} \cup \{1, 7\} \cup \{2, 8\} \cup \{3, 9\} \cup \{4, 10\} \cup \{5, 11\} = Z_{12} = \underline{I^*(A \cup B)}$,

$\overline{I^*(A)} \cup \overline{I^*(B)} = \{1, 2, 3, 4, 5, 7, 8, 9, 10, 11\} \subseteq Z_{12}$.

In a similar way we can prove approximations of sets $(A \cap C)$ and $(A \cup C)$.

If $D \subseteq A$ s.t $D = \{1, 3, 5, 7, 9, 11\} \subseteq \mathrm{A}$ then $\underline{I^*(D)} = \{x^* \in Z_{12}: (x^* + I^*) \subseteq D\} = \{1, 7\} \cup \{3, 9\} \cup \{5, 11\} = \{1, 3, 5, 7, 9, 11\}\} \subseteq \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\} = \underline{I^*(A)}$. Also $\overline{I^*(D)} = \{x^* \in Z_{12} : (x^* + I^*) \cap D \neq \emptyset\} = \{1, 3, 5, 7, 9, 11\} \subseteq Z_{12} = \overline{I^*(A)}$.

Now Since $B \cap C = \emptyset$ then $\underline{I^*(B \cap C)} = \{x^* \in Z_{12}: (x^* + I^*) \subseteq (B \cap C)\} = \emptyset$,

$\underline{I^*(B)} \cap \underline{I^*(C)} = \{2, 4, 8, 10\} \cap \{5, 11\} = \emptyset$. There for $\underline{I^*(B \cap C)} = \underline{I^*(B)} \cap \underline{I^*(C)}$, $\overline{I^*(B \cap C)} = \{x^* \in Z_{12} : (x^* + I^*) \cap (B \cap C) \neq \emptyset\} = \emptyset$,

$\overline{I^*(B)} \cap \overline{I^*(C)} = \{1, 7\}$ that is $\overline{I^*(B \cap C)} = \emptyset \subseteq \overline{I^*(B)} \cap \overline{I^*(C)}$.

Also since $B \cup C = \{1, 2, 4, 5, 6, 7, 8, 9, 10, 11\}$,

$\underline{I^*(B \cup C)} = \{x^* \in Z_{12}: (x^* + I^*) \subseteq (B \cup C)\} = \{1,7\} \cup \{2,8\} \cup \{4,10\} \cup \{5,11\} = \{1,2,4,5,7,8,10,11\}$.

$\underline{I^*(B)} \cup \underline{I^*(C)} = \{2,4,5,7,8,10,11\} \subseteq \underline{I^*(B \cup C)}$.

$\overline{I^*(B \cup C)} = \{x^* \in Z_{12} : (x^* + I^*) \cap (B \cup C) \neq \emptyset\} = \{0,6\} \cup \{1,7\} \cup \{2,8\} \cup \{3,9\} \cup \{4,10\} \cup \{5,11\} = \{0,1,2,3,4,5,6,7,8,10,11\} = Z_{12}$,

$\overline{I^*(B)} \cup \overline{I^*(C)} = \{0,1,2,4,6,7,8,10\} \cup \{1,3,5,7,9,11\} = Z_{12}$. So $\overline{I^*(B \cup C)} = \overline{I^*(B)} \cup \overline{I^*(C)}$

Since $A^c = \{0\}$ then $\underline{I^*(A^c)} = \{x^* \in Z_{12}: (x^* + I^*) \subseteq A^c\} = \emptyset$, $(\underline{I^*(A^c)})^c = Z_{12} = \overline{I^*(A)}$,

$\overline{I^*(A^c)} = \{x^* \in Z_{12}: (x^* + I^*) \cap A^c \neq \emptyset\} = \{0,6\}$,

$(\overline{I^*(A^c)})^c = \{1,2,3,4,5,6,7,8,9,10,11\} = \underline{I^*(A)}$

$\underline{I^*\left(\underline{I^*(A)}\right)} = \left\{x^* \in Z_{12}: (x^* + I^*) \subseteq \underline{I^*(A)}\right\} = \{1,2,3,4,5,6,7,8,9,10,11\} = \underline{I^*(A)}$.

$\overline{I^*\,(\underline{I^*(A)})} = \left\{x^* \in Z_{12}: (x^* + I^*) \cap \underline{I^*(A)} \neq \emptyset\right\}$
$= \{1,7\} \cup \{2,8\} \cup \{3,9\} \cup \{4,10\} \cup \{5,11\}$

$= \{1,2,3,4,5,6,7,8,9,10,11\} = \underline{I^*(A)}$,

$\overline{I^*(\overline{I^*(A)})} = \left\{x^* \in Z_{12}: (x^* + I^*) \cap \overline{I^*(A)} \neq \emptyset\right\}$

$= \{0,6\} \cup \{1,7\} \cup \{2,8\} \cup \{3,9\} \cup \{4,10\} \cup \{5,11\} = Z_{12} = \overline{I^*(A)}$,

$\underline{I^*(\overline{I^*(A)})} = \left\{x^* \in Z_{12}: (x^* + I^*) \subseteq \overline{I^*(A)}\right\} = Z_{12} = \overline{I^*(A)}$.

Proposition 4.3: Let ($\Re^*$, mod$A$) be a* - ring approximation space, and I^*, J^* are two * - ideals of $\Re^*$, Then $\overline{I^*\,(J^*)}$ and $\underline{I^*\,(J^*)}$ are * - ideals of $\Re^*$.

Proof: we need to prove $\overline{I^*\,(J^*)}$ and $\underline{I^*\,(J^*)}$ is closed under involution *. Since I^* and J^* are * - ideals then $I^* \subseteq I$ and $J^* \subseteq J$. So $I^* \cap J^* \subseteq I \cap J \neq \emptyset$, then there exist $x^* \in I^* \cap J^*$.

But $I^* \subseteq x^* + I^*$. So we get $(x^* + I^*) \cap J^* \neq \emptyset$, that is $x^* \in \overline{I^*\,(J^*)}$. Subsequently $\overline{I^*\,(J^*)}$ is * - ideal of $\Re^*$. In a similar way we can prove $\underline{I^*\,(J^*)}$ is * - ideal of $\Re^*$■

The following example shows Proposition 4.3.

Example 4.3: Let the ring $\Re^* = \mathrm{Z}_{12}$ and define the involution on $\Re^*$ by $a^* = a \ \ \forall a \in \mathrm{Z}_{12}$. Suppose a * - ideals are $I^* = \{0, 3, 6, 9\}$and $J^* = \{0, 6\}$. Since the involution on $\Re^*$ define by

$a^* = a \ \ \forall a \in \Re$, Then $I^* = I$; $I*$ is a * - ideal. For $x^* \in \Re^*$: $x^* + I^*$, it can get {0*, 3*, 6*, 9*} = {0, 3, 6, 9}, {1*, 4*, 7*, 10*} = {1, 4, 7, 10}, {2*, 5*, 8*, 11*} = {2, 5, 8, 11}. Then, the lower approximation of J^* with respect of I^* as:

$\underline{I^*\,(J^*)} = \{x^* \in \Re^* : (x^* + I^*) \subseteq J^*\} = \emptyset$ is a traivel $*-$ideal and the upper approximation of

J^* with respect of I^* as:

$$\overline{I^*\,(J^*)} = \cup \{x^* \in \Re : (x^* + I^*) \cap J^* \neq \emptyset\} = \overline{I^*\,(J^*)}$$

$$= \{x \in \Re : (x + I) \cap J^* \neq \emptyset\} = \{0, 3, 6, 9\} \text{ is * - ideal in } \mathrm{Z}_{12} .$$

Subsequently $\overline{I^*\,(J^*)}$ and $\underline{I^*\,(J^*)}$ are * - ideal of $\Re^*$.

5. Rough * - ideals in a ring with involution:

In this section, we introduce the concept of rough * - ideal in ring with involution and give some result on them.

Definition 5.1: Let I^* be * - ideal of involution ring $\Re^*$ and X is rough set with respect to I^* , we have: (i) If $\underline{I^*}(X)$ and $\overline{I^*}(X)$ are two $*$ - ideals of $\Re^*$, then we call X is a rough $* -$ ideal.

(ii) If $\underline{I^*}(X)$ and $\overline{I^*}(X)$ are an involution subring of $\Re^*$, then we call X is a rough involution ring. The following example shows definition 5.1.

Example 5.1: We consider the ring $\Re^* = Z_8$ and the ideal $I = \{0, 2, 4, 6\}$.

we define the involution on $\Re^*$ by a * = a $\forall$ a $\in Z_8$. There for a * - ideal is {0, 2, 4, 6} and let

$X = \{0, 1, 2, 4, 6\}$, $Y = \{1, 2, 3, 4, 5, 6, 7\}$ Since the involution on Z_8 define by a * = a $\forall$ a $\in \Re$, Then $I^* = I$; I^* is a * - ideal. For $x^* \in \Re^*$: $x^* + I^*$, it can get {0, 2, 4, 6}, {1, 3, 5, 7}. There for

$\underline{I^*}(X) = \{x^* \in \Re^* : (x^* + I^*) \subseteq X\} = Z_8$ is a trivel * - ideal Subsequently ideal in Z_8 and

$\overline{I^*}(X) = \cup \{x^* \in \Re^* : (x^* + I^*) \cap X \neq \emptyset\} = \{0, 2, 4\}$ is $* -$ ideal in Z_8 Subsequently ideal in Z_8 ;$I * (X) \subseteq I(X)$; $I * \subseteq I$. So by definition 5.1 ($\underline{I^*}(X)$, $\overline{I^*}(X)$) are a rough * - ideals. Subsequently rough ideal in Z_8 Not that $\overline{I^*}(X)$is sub ring in Z_8 and is not ideal Now when because 7(2) = 6 mod (8) and 6 $\notin \overline{I^*}(X)$. There for ($\underline{I^*}(X)$, $\overline{I^*}(X)$) are an involution subring of $\Re^*$ and X is a rough an involution ring. Subsequently are a subring of $\Re^*$ and X is a rough a ring. Now if $Y = \{1, 2, 3, 4, 5, 6, 7\}$ then we have: $\underline{I^*}(Y) = \{1, 3, 5, 7\}$ and $\overline{I^*}(Y) = Z_8$ is a travel $* -$ ideal . Not that $\underline{I^*}(Y)$ is $* -$ ideal in Z_8 Because $\underline{I^*}(Y)$ closed under involution; $\underline{I^*}(Y) = \underline{I}(Y)$. but not ideal because $\forall r \in Z_8 \wedge \forall a \in \underline{I}(Y)$ we find that $0 \in Z_8 \wedge 3 \in \underline{I^*}(Y)$. So

0 (3) = 0 $\notin$ $\underline{I^*(Y)}$; $\underline{I^*(Y)} = \underline{I(Y)}$. As well $\underline{I^*(Y)}$is not subring because 3 – 1= 2 $\notin$ $\underline{I^*(Y)}$. But

$\underline{I^*(Y)}$ is not an involution sub ring in Z_8. There for $\left(\underline{I^*(Y)},\ \overline{I^*(Y)}\right)$ are not an involution subring of Z_8 Subsequently are not a subring of Z_8 and Y is not a rough an involution ring. Therefor $\left(\underline{I^*(Y)},\ \overline{I^*(Y)}\right)$ is not a rough involution ring.Subsequently Y is not a rough a ring.

Proposition 5.1: let ($\Re$*, mod$A$) be a * -ring approximation space and I*, J* be two * - ideals of involution ring $\Re$*, Then

(i) $\overline{I^*(J^*)}$ and $\underline{I^*(J^*)}$ are rough * - ideals of $\Re$* .
(ii) Let I^* is * - ideal and J^* is *- subring of involution ring $\Re$* , Then $\overline{I^*(J^*)}$ and $\underline{I^*(J^*)}$ are an involution rings.

Proof: (i): Since I^*, J^* be two * - ideals of involution ring $\Re$* , Then by Proposition 4.2 $\overline{I^*(J^*)}$ and $\underline{I^*(J^*)}$ are rough * - ideals of $\Re$* . So by definition 4.1 J* is rough * - ideals with respect of I*. And Since ($\underline{I^*(J^*)}$, $\overline{I^*(J^*)}$) are * - ideals then $\underline{I^*(J^*)} = \{x^* \in \Re^* : (x^* + I^*) \subseteq X\}$ and $\overline{I^*(J^*)} = \{x^* \in \Re^* : (x^* + I^*) \cap X \neq \emptyset\}$ are a lower and upper with respect of I*, respectively. So ($\underline{I^*(J^*)}$, $\overline{I^*(J^*)}$) are rough * - ideals of $\Re$*.

(ii) Since I* a * - ideal then I* is a *- subring of involution ring $\Re$*, But J * not a * - ideals of involution ring $\Re$*. From (i) $\overline{I^*(J^*)}$ and $\underline{I^*(J^*)}$ are rough * - ideals of $\Re$* Then $\overline{I^*(\overline{I^*(J^*)})}$ and $\underline{I^*(\overline{I^*(J^*)})}$ are a rough * - ideals. And so $\overline{I^*(J^*)}$ and $\underline{I^*(J^*)}$ are an involution rings■

6. CONCLUSION:

The object of this paper is to introduce the lower and upper approximations of any subset of a ring with involution which we called * - ring approximation spaces respect to * -ideals. Some basic properties of these operators were presented. Rough * - ideals are introduced in * - ring approximation spaces. Our research in this area is still on going. We are currently in the midst of extending the study of * - ring approximation spaces with some topological concepts. Additional future research also includes a deeper study of * - ring approximation spaces in neutrosophic topology.

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